19. If the elements S and T in $A(V)$ are similar in $A(V)$, prove that they have the same elementary divisors.

20. For all $S, T \in A(V)$, λ in F, show that :

(a) $(S+T)^* = S^* + T^*$

(b) $T^{**} = T$

(c) $(ST)^* = T^* S^*$.

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NOVEMBER/DECEMBER 2024

23PMA11 — ALGEBRAIC STRUCTURES

Time : Three hours Maximum : 75 marks

SECTION A — ($10 \times 2 = 20$ marks)

Answer ALL questions.

1. Prove that if G is finite, $a \in z$ if and only if $O(N(G)) = O(G)$.

2. Define conjugate of an element in G.

3. When we say that two transformations are similar.

4. Define index of nilpotent of T.

5. Define invariants of an abelian group.

6. Define finitely generated of a module over R.

7. Define elementary divisors of polynomials of T.

8. Define companion matrix of $f(x) \in F[x]$.

9. Show that $tr(AB) = tr(BA)$ if $A, B \in Fn$, $\lambda \in F$.

10. Define hermitian adjoint of T.

SECTION B — (5 × 5 = 25 marks)

Answer ALL questions.

11. (a) Prove that conjugacy is an equivalence relation on G.

Or

(b) Show that $O(AxB) = \dfrac{O(A)O(B)}{O(A \cap xBx^{-1})}$, if A, B are finite subgroups of G.

12. (a) If G and G' are isomorphic abelian groups, show that for every integer S, $G(s)$ and $G(S')$ are isomorphic.

Or

(b) Show that G is solvable if and only if $G^{(K)} = e$ for some integer K.

13. (a) Let W be invariant under T and let T induces $\overline{T} : \dfrac{V}{W} \to \dfrac{V}{W}$ by $\overline{T}(V + W)$. Prove that if T satisfies $q(x) \in F[x]$ so does $\overline{T}$.

Or

(b) If $T \in A(V)$ is nilpotent, prove that $\alpha_0 + \alpha_1 T + ... + \alpha_m T^m, \alpha_i \in F$ is invertible, $\alpha_0 \neq 0$.

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14. (a) Prove that every $T \in A(V)$ satisfies its characteristic polynomial.

Or

(b) If $A, B \in F_n$ and A and B are similar in K_n, where K is an extension of F, prove that A and B are similar in F_n.

15. (a) If $\langle T(v), T(v) \rangle = \langle v, v \rangle$, $\forall v \in V$, prove that T is unitary.

Or

(b) If N is normal and if $N^k(v) = 0$, show that $N^k v = 0$.

SECTION C — (3 × 10 = 30 marks)

Answer any THREE questions.

16. If P is a prime number and $P \mid O(a)$, prove that G has an element of order P.

17. Show that $G^{(K)}, K = 1, 2,...$ contains 3-cyde of S_n, where $G = S_n, n \geq 5$.

18. Prove that there is a subspace W of V invariant under T such that $V = V_1 \oplus W$.